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Notes, Comments, and Letters to the Editor

Stochastic Dynamic Models with Stock-Dependent Rewards*

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We examine the behavior of optimal consumption and investment policies in aggregate stochastic growth models when utility depends on both consumption and the stock level. Such models arise in the study of renewable resources, monetary growth, and growth with public capital. Conditions are given which guarantee that optimal policies are monotonic. The limiting behavior of the optimal consumption, investment, and output processes is characterized. *Journal of Economic Literature* Classification Numbers: 026, 111, 721.

1. INTRODUCTION

For a number of important dynamic resource allocation problems, the reward or utility of an agent depends on the size of the resource stock as well as the amount consumed in any period. This is perhaps most evident in renewable resource models but there are other problems where stock-dependent rewards are important as well. These include the effect of wealth on consumption and savings behavior [15], the role of money in a growing economy [4, 7, 26], liquidity constraints and the incorporation of real balances in the utility function [11], and the theory of the balance of payments in dual exchange rate economies [12, 24].

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Much of the literature on stochastic dynamic resource allocation focuses on the long run behavior of optimal consumption and investment policies. For the one-sector optimal growth model with stock-independent rewards and concave production, a series of papers by Brock and Mirman [5, 6] and Mirman and Zilcha [21, 22] examine conditions under which the optimal capital stock converges to a *unique* limiting distribution. These results are extended to a model with non-convex technology by Majumdar, Mitra, and Nyarko [18] and to the case of irreversible investment by Olson [25].

The purpose of this note is to investigate the monotonicity and convergence properties of optimal policies within the context of a generic stochastic growth model with stock-dependent rewards. In our investigation we derive most results strictly from assumptions imposed on the primitive data of the model, that is, assumptions on preferences and technology. In Section 2, we show that complementarity between state and policy variables in the reward function is sufficient for optimal consumption and investment policies to be nondecreasing functions of the resource stock. We then investigate the dynamic behavior of the model and characterize the convergence of optimal resource stocks. Proofs are given in Section 3.

There is a close relationship between our work and the analysis of renewable resource markets by Mendelssohn and Sobel [19] and Mirman and Spulber [20] in the case of uncertainty, and by Levhari, Michener, and Mirman [13, 14] in a deterministic setting. Unpublished work by Majumdar [17] also examines a deterministic version of the problem. A related, but alternative approach to studying convergence is given in Easley and Spulber [10].

2. Results

At each date t there is a resource stock denoted by $y_t \in \mathscr{R}_+$. Given knowledge of y_t, the agent determines a consumption level c_t. The resource stock left at the end of date t (after consumption) represents investment and is denoted by $x_t = y_t - c_t$. Let $\{r_t\}_{t=1}^{\infty}$ be an independent and identically distributed (iid) process taking values in some compact set Φ, where Φ is a subset of a finite dimensional Euclidean space. Growth in the resource stock is governed by the production relationship, $y_{t+1} = f(x_t, r_{t+1}) = f(y_t - c_t, r_{t+1})$, where $f\colon \mathscr{R}_+ \times \Phi \to \mathscr{R}_+$ is the production function. Let γ be the (common) probability measure associated with the shock process r_t. At the beginning of each period (before the consumption decision), it is assumed that the agent observes the true resource stock.

Given a resource stock, y_t, and a consumption, c_t, the agent receives rewards $R(c_t, y_t)$. Given an initial stock, $y_0 > 0$, the agent seeks to solve

$$V(y_0) = \max_{\{c_t\}} E \sum_{t=0}^{\infty} \delta^t R(c_t, y_t) \tag{2.1}$$

subject to $0 \leqslant c_t \leqslant y_t$, where $\delta \in (0, 1)$ is the discount factor and V is the value function.

The production and reward functions are assumed to satisfy the following restrictions.

A.2.1. For all r, $f(x, r)$ is strictly increasing in x.

A.2.2. f is concave in x.

A.2.3. For all r, $f(0, r) = 0$ while $f(x, r) > 0$ if $x > 0$.

A.2.4. The first and second derivatives of $f(x, r)$ in x exist and are continuous in (x, r).

A.2.5. There exists a $\bar{y}$ such that $f(y, r) < y$ a.s. for all $y \geqslant \bar{y}$.

A.2.6. $R(c, y)$ is nondecreasing in y.

A.2.7. $R(c, y)$ is concave in (c, y).

A.2.8. $R(c, y)$ is twice continuously differentiable.

A.2.9. $y_0 \in (0, \bar{y}]$.[1]

A.2.10. $f(x, r)$ is strictly concave in x for each r and $R(c, y)$ is strictly increasing in y.

A.2.10′. $R(c, y)$ is strictly concave.

The specification of the model satisfies the usual continuity and boundedness conditions so that standard dynamic programming arguments (cf. Blackwell [3] and Maitra [16]) can be used to show that there exists a stationary optimal policy function of the form $c_t = C^*(y_t)$. This gives the following result.

THEOREM 2.1. *Under* A.2.1–A.2.10 *there exists a stationary optimal policy function* $c_t = C^*(y_t)$. *Furthermore, the following functional equation holds*:

$$V(y) = \operatorname*{Max}_{0 \leqslant c \leqslant y} R(c, y) + \delta \int V(f(y - c, r))\, \gamma(dr). \tag{2.2}$$

In addition, the value function is continuous and the optimal policy function is continuous.

[1] This assumption is without loss of generality due to A.2.5.

Given $C^*(y)$, the optimal investment policy function is defined to be $X^*(y) = y - C^*(y)$. Define y_t^*, c_t^*, and x_t^* to be the optimal output stock, consumption level, and investment at date t. Henceforth, we assume that the optimal consumption and investment policies are strictly positive. This will hold if the usual Inada conditions are imposed on R. Under this assumption, one can show that optimal processes are characterized by the following stochastic Euler equation, where R_c and R_y denote the derivatives of R with respect to c and y.

THEOREM 2.2. *Under* A.2.1–A.2.10, *if* $0 < c_t^* < y_t^*$ *for all* t, *then* y_t^*, c_t^*, x_t^* *satisfy*

$$R_c(c_t^*, y_t^*) = \delta E\{[R_c(c_{t+1}^*, y_{t+1}^*) + R_y(c_{t+1}^*, y_{t+1}^*)] f'(x_t^*, r_{t+1})\}. \tag{2.3}$$

We now examine the monotonicity properties of the optimal consumption and investment policies under the following assumptions.

A.2.11. $R_{cy} \geqslant 0$.

A.2.11′. $R_{cy} > 0$.

A.2.12. $R_{cc} + R_{cy} \leqslant 0$.

A.2.12′. $R_{cc} + R_{cy} < 0$.

A.2.11 and A.2.12 can be interpreted as complementarity conditions between consumption and output from production on the one hand, and investment and output from production on the other.[2]

THEOREM 2.3. *Under* A.2.11, $C^*(y)$ *is nondecreasing in* y. $C^*(y)$ *is strictly increasing if* A.2.11′ *holds.*

THEOREM 2.4. *Under* A.2.12, $X^*(y)$ *is nondecreasing in* y. $X^*(y)$ *is strictly increasing if* A.2.12′ *holds.*

Theorem 2.3 extends similar results obtained by Brock and Mirman [5], Mirman and Zilcha [21], and Mendelssohn and Sobel [19], all of whom assume that the reward function depends solely on consumption. Theorem 2.4 is the stochastic analogue of deterministic results obtained by Majumdar [17] and Benhabib and Nishimura [1], and is similar to Theorem 4.2 in Mendelssohn and Sobel [19].

We now study the limiting behavior of the optimal stochastic investment process governed by the transition equation $x_{t+1} = X^*(f(x_t, r_{t+1}))$. The

[2] If $R(c, y)$ is expressed as $R(y - x, y) = W(x, y)$, then $W_x = -R_c$ and $R_{cc} + R_{cy} \leqslant 0$ implies $W_{xy} = -R_{cc} - R_{cy} \geqslant 0$.

monotonicity and continuity of the optimal investment and consumption policy functions are used to prove that optimal processes converge to an invariant distribution, the stochastic analogue of a steady state. Then, we give sufficient conditions for the invariant distribution to be unique. In general, however, the invariant distribution need not be unique. The analysis relies on techniques developed by Dubins and Freedman [9] to study the convergence behavior of the Markov process. In the economics literature, these methods were first used by Majumdar, Mitra, and Nyarko [18] to study the stochastic optimal growth model with a non-convex technology.

Define $r^t = (r_1, ..., r_t)$ and let γ^t be the joint distribution of r^t. For each n and r^n, define $X^n(\cdot, r^n)$ by the relation $X^n(x_0, r^n) = X^*(f(\cdots X^*(f(X^*(f(x_0, r_1)), r_2)), ..., r_n))$ so that $X^n(x_0, r^n)$ is the realization of x_n given x_0 and $r^n = (r_1, ..., r_n)$. If μ is any probability on $\mathscr{R}_+$ define the probability $\gamma^n\mu$ on $\mathscr{R}_+$ to be $\gamma^n\mu(A) = \int \gamma^n(\{r^n \mid X^n(x_0, r^n) \in A\})\, \mu(dx_0)$, where A is any (Borel) subset of $\mathscr{R}_+$. $\gamma^n\mu$ is the distribution of x_n when the distribution of x_0 is μ. μ is an *invariant probability* if $\gamma^1\mu = \mu$. A subset S' of $\mathscr{R}_+$ is said to be *γ-invariant* if it is closed and if $\gamma(\{r \in \Phi \mid X^*(x, r) \in S'$ for all $x \in S'\}) = 1$. A subset S'' is a *minimal γ-invariant set* if it is γ-invariant and if any strict subset of S'' is not γ-invariant.

Define $f_m(x) = \min_r f(x, r)$, $f_M(x) = \max_r f(x, r)$, $X_m(x) = X^*(f_m(x))$, and $X_M(x) = X^*(f_M(x))$. The minimum and maximum are well defined since f and X^* are continuous and defined over the compact domain $[0, \bar{y}]$, where $\bar{y}$ is given in A.2.5. Further, from the Maximum Theorem [2, p. 116], f_m, f_M, X_m, and X_M are all continuous in x. Let S be some γ-invariant interval $[\underline{s}, \bar{s}]$. We assume that on S, $X_m(x) < X_M(x)$.[3] Define the sequence $\{a_n, \hat{a}_n, b_n\}_{n=1}^{\infty}$ inductively by $a_n = \mathrm{Inf}\{x \geq b_{n-1} \mid X_m(x) = x\}$, $b_n = \mathrm{Inf}\{x > a_n \mid X_M(x) = x\}$, $\hat{a}_n = \mathrm{Sup}\{x \in [a_n, b_n] \mid X_m(x) = x\}$, where $b_0 = \underline{s}$ (see Fig. 1). Since X_m and X_M are continuous and S is compact, Brouwer's fixed point theorem implies that X_m and X_M have at least one fixed point. Thus, a_n, $\hat{a}_n$, and b_n are well-defined for at least one n. Let N^* be the maximum over n such that a_n, $\hat{a}_n$, and b_n are well-defined. We can now state the following convergence result.

THEOREM 2.5. (a) *Fix any integer $n \in [1, N^*]$. Then the set $[\hat{a}_n, b_n]$ is a γ-invariant interval and $[\hat{a}_n, b_n]$ is a unique γ-invariant interval in itself. There exists a unique γ-invariant distribution on $[\hat{a}_n, b_n]$. If $x_0 \in [\hat{a}_n, b_n]$ then the distribution function of x_t^* converges uniformly to the distribution function of the invariant distribution on $[\hat{a}_n, b_n]$.*

[3] Let $\bar{y}$ be as in A.2.5. Under an Inada condition on f it can be shown that there exists an $\varepsilon > 0$ such that $[\varepsilon, \bar{y}]$ is γ-invariant. Further, if there exists no $x > 0$, $y \geq 0$ such that $\gamma(\{r \mid f(x, r) = y\}) = 1$, then $X_m(x) < X_M(x)$ on $[\varepsilon, \bar{y}]$ (see Nyarko and Olson [23] for details).

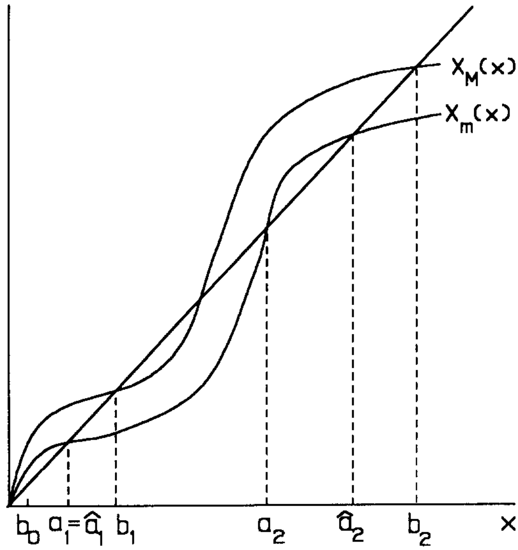

FIG. 1. Illustration of fixed points a_n, $\hat{a}_n$, and b_n.[4]

(b) *The set* $T=\bigcup_{n=2}^{N^*}(b_{n-1}, \hat{a}_n) \cup [b_0, \hat{a}_1) \cup (b_{N^*}, \bar{y}]$ *is transient. If* $x_0 \in T$, *then, with probability one,* x_t^* *will in finite time leave* T *and never return.*

(c) *There is a unique invariant probability on* S *if and only if* $N^* = 1$.

This theorem states that once an optimal process enters an interval $[\hat{a}_n, b_n]$ it will remain there forever and converge to a unique limiting distribution on $[\hat{a}_n, b_n]$. Furthermore, sets of the type $(b_{n-1}, \hat{a}_n)$ are transient. Optimal processes converge globally to the same (unique) invariant distribution on S if and only if there is only one interval of the type $[\hat{a}_n, b_n]$.

3. PROOFS

Proof of Theorem 2.1. See Nyarko and Olson [23]. ▌

Proof of Theorem 2.2. The proof follows the proof of Theorem 4.2 in Majumdar, Mitra, and Nyarko [18] once it is modified to account for the dependence of R on both c and y. ▌

Proof of Theorems 2.3. and 2.4. With suitable modifications, the proofs follow arguments originally given in Dechert and Nishimura [8]. Complete proofs are given in Nyarko and Olson [23]. ▌

[4] In Fig. 1, the space S could be taken to be the interval $[b_0, b_2]$.

Proof of Theorem 2.5. (a) Fix any integer n in $[1, N^*]$ and any x in $[\hat{a}_n, b_n]$. Then for any r in Φ, $\hat{a}_n = X_m(\hat{a}_n) \leqslant X_m(x) \leqslant X^*(f((x, r)) \leqslant X_M(x) \leqslant X_M(b_n) = b_n$ so for all r, $X^*(f(x, r)) \in [\hat{a}_n, b_n]$. The set $[\hat{a}_n, b_n]$ is therefore a γ-invariant closed interval.

Now let $[a, b]$ be any closed interval in $[\hat{a}_n, b_n]$. If $b < b_n$, $X_M(b) > b$, so $[a, b]$ is not γ-invariant. If $a > a_n$ then $X_m(a) < a$, so $[a, b]$ is not γ-invariant. Hence there is no closed interval $[a, b]$ strictly contained in $[\hat{a}_n, b_n]$ which is γ-invariant so $[\hat{a}_n, b_n]$ is the unique (and hence minimal) γ-invariant interval in itself. The uniform convergence of the distribution of x_t^* to the γ-invariant distribution on $[\hat{a}_n, b_n]$ follows from Dubins and Freedman [9, Theorem 5.15 and Corollary 5.5].

(b) Fix any integer n in $[2, N^*]$. For such n, the set $(b_{n-1}, \hat{a}_n)$ has the two intervals $[\hat{a}_{n-1}, b_{n-1}]$ and $[\hat{a}_n, b_n]$ to its left and right, respectively, and both of these sets are γ-invariant as shown in (a) above. Hence, once the x_t process leaves the set $(b_{n-1}, \hat{a}_n)$ it never returns to it.

Note that $X_M(x) > x$ for all x in $[a_n, \hat{a}_n)$. Since X_M is continuous there exists an $\varepsilon > 0$ and a $k > 0$ such that $X_M(x) > x + k$ for $x \in [a_n - \varepsilon, \hat{a}_n)$. Define the sequence of numbers $\{\hat{x}_t\}_{t=0}^{\infty}$ by $\hat{x}_0 = a_n - \varepsilon$ and $\hat{x}_{t+1} = X_M(\hat{x}_t)$. Let J_1 be any integer greater than $(\hat{a}_n - (a_n - \varepsilon))/k$. Then it is easy to see that $\hat{x}_t$ will enter the set $[\hat{a}_n, b_n]$ in less than J_1 periods.

For any $\{x_t\}_{t=0}^{\infty}$ process generated by $X^*(f(x, r))$, define $q_1 = \text{Prob}(\{x_{J_1} \in [\hat{a}_n, b_n] \mid x_0 = a_n - \varepsilon\})$. Then $q_1 > 0$. Since the transition function $X^*(f(x, r))$ is monotone in x, $\text{Prob}(\{x_{J_1} \in [\hat{a}_n, b_n] \mid x_0 \in [a_n - \varepsilon, \hat{a}_n)\}) \geqslant q_1$. Since $X_m(x) < x$ for all x in $[b_{n-1}, a_n - \varepsilon)$ there exists a $q_2 > 0$ and an integer J_2 such that $\text{Prob}(\{x_{J_2} \in [\hat{a}_{n-1}, b_{n-1}] \mid x_0 \in (b_{n-1}, a_n - \varepsilon)\}) \geqslant q_2$. Define $q = \text{Min}\{q_1, q_2\}$ and $J = J_1 + J_2$. Then, $\text{Prob}(\{x_J$ leaves $(b_{n-1}, \hat{a}_n) \mid x_0 \in (b_{n-1}, \hat{a}_n)\}) \geqslant q > 0$. Finally, if x_0 belongs to $(b_{n-1}, \hat{a}_n)$, $\text{Prob}(\{x_t \in (b_{n-1}, \hat{a}_n)$ for all $t\}) \leqslant \text{Prob}(\{x_{kJ} \in (b_{n-1}, \hat{a}_n)$ for all $k = 1, 2, \ldots\}) = \prod_{k=1}^{\infty} \text{Prob}(\{x_{kJ} \in (b_{n-1}, \hat{a}_n) \mid x_{mJ} \in (b_{n-1}, \hat{a}_n)$ for all $m < k\}) \leqslant \prod_{k=1}^{\infty} (1 - q) = 0$. Hence, the set $(b_{n-1}, \hat{a}_n)$ for $2 \leqslant n \leqslant N^*$ is transient.

Using similar methods one can show that if $x_0 \in [b_0, \hat{a}_1)$ then x enters $[\hat{a}_1, b_1]$ in finite time and if $x_0 \in (b_{N^*}, \bar{y}]$ then x_t enters $[\hat{a}_{N^*}, b_{N^*}]$ in finite time. This proves part (b). Part (c) follows immediately from (a) and (b). ∎

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